THE APPLICATION OF INTEGRAL EQUATIONS TO THE PROBLEM OF TORSION OF SHAFTS OF VARIABLE DIAMETER

(PRIMENENIE INTEGRAL'NYKH URAVNENII V ZADACHE O KRUCHENII VALOV PEREMENNOGO DIAMETRA)

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1. Formulation of the problem. We shall take a rod (shaft) of variable section to mean a body formed by the rotation of some plane curve L about an axis lying in the same plane as the curve.

Let us consider a system of cylindrical coordinates r, z, ϕ with the z-axis as the axis of the shaft. Let us suppose that the load applied to the body is distributed symmetrically about the z-axis and that it acts in a direction perpendicular to the plane $\phi = \text{const}$. In this case the displacements of points of the shaft will also be distributed symmetrically, i.e. they will be independent of ϕ. We can assume that the displacements $v(r, z)$ take place in a direction perpendicular to the plane $\phi = \text{const}$.

With these assumptions the equations of elastic equilibrium in terms of displacements reduce to the single equation

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (rv)}{\partial r}\right) + \frac{\partial^2 v}{\partial z^2} = 0 \tag{1.1}$$

Equation (1.1) is closely related to the Laplace equation given in [1]. Putting $u(r, z, \phi) = v(r, z)\, e^{i\phi}$ we have $\Delta u = 0$.

With stresses given on the surface of the shaft, in order to simplify the boundary condition it is expedient to replace the function $v(r, z)$ by a stress function $\Phi(r, z)$ related to $v(r, z)$ by the expressions

$$\mu \frac{\partial}{\partial r}\frac{v}{r} = -\frac{1}{r^3}\frac{\partial \Phi}{\partial z}, \quad \mu \frac{\partial v}{\partial z} = \frac{1}{r^2}\frac{\partial \Phi}{\partial r} \qquad \mu \text{ is the shear modulus} \tag{1.2}$$

The function $\Phi(r, z)$ satisfies the equation

$$\frac{\partial^2\Phi}{\partial r^2} - \frac{3}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{1.3}$$

The stresses can be expressed in terms of $\Phi(r, z)$ by the formulas

$$\tau_{r\varphi} = -\frac{1}{r^2}\frac{\partial\Phi}{\partial z}, \qquad \tau_{z\varphi} = \frac{1}{r^2}\frac{\partial\Phi}{\partial r} \tag{1.4}$$

The remaining components of the stress tensor are zero.

On the boundary L the values of $\Phi(r, z)$ can easily be found from the known values of the stresses

$$\Phi(r, z) = \int_0^s \left(\frac{\partial\Phi}{\partial z}\frac{dz}{ds} + \frac{\partial\Phi}{\partial r}\frac{dz}{ds}\right) ds = \int_0^s \left(\tau_{z\varphi}\frac{dr}{ds} - \tau_{r\varphi}\frac{dz}{ds}\right)\frac{ds}{r^2} + C = f(s) + C \tag{1.5}$$

Here $f(s)$ is a given function of the arc length s of the boundary L, and C is an arbitrary constant.

2. The general solution of Equation (1.3) and the reduction of the boundary problem to an integral equation. 1. Let us assume that

$$\Phi(r, z) = r^2 w(r, z) \tag{2.1}$$

The function $w(r, z)$ satisfies the equation

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{4}{r^2}w + \frac{\partial^2 w}{\partial z^2} = 0 \tag{2.2}$$

Equation (2.2), which is related to the Laplace equation $\Delta\{w(r, z)e^{2i\phi}\} = 0$, is considered in [1]. The general solution to Equation (2.2) is given by the integral [2]

$$w(r, z) = \mathrm{Re}\int_0^\pi \psi(r\cos\lambda + iz)\cos 2\lambda d\lambda \tag{2.3}$$

Here $\psi(\xi + iz)$ is a function, regular within the area formed by an axial section of the shaft (it is assumed that any line perpendicular to the z-axis intersects the boundary L at one point only); also, the function ψ must satisfy the requirement that $\mathrm{Re}\,\psi(\xi + iz) = \mathrm{Re}\,\psi(-\xi + iz)$, i.e. the requirement that it is even with respect to r.

(2.3) is not a unique solution; any linear function $ar + b$ can be added to $\psi(t)$ (which was not pointed out in [1]).

After integrating twice by parts we can express $w(r, z)$ as an integral which satisfies the uniqueness theorem

$$w(r,\ z) = r^2 \int_0^{\pi} \operatorname{Re} \chi (r \cos \lambda + iz) \sin^4 \lambda d\lambda \qquad (\chi(\tau) = \psi''(\tau)) \tag{2.4}$$

Thus

$$\Phi(r,\ z) = r^4 \int_0^{\pi} \operatorname{Re} \chi (r \cos \lambda + iz) \sin^4 \lambda d\lambda \tag{2.5}$$

If we now put $\chi = \text{const}$ we obtain the solution to the problem of a circular cylindrical shaft subjected to a torque applied at infinity.

We can therefore try to find a solution to the problem of torsion in a cylindrical notched shaft in the form (2.5), putting $\chi(\tau) = c + \chi_0(\tau)$, where $\chi_p(\tau)$ is a function which vanishes at infinity.

In solving a problem on torsion in an infinitely long shaft with different diameters at opposite ends we assume that

$$\chi(\tau) = ic \ln(\tau_1 / \tau_2) + \chi_0(\tau) + c_1 \qquad (\tau_1 = \tau - \tau_0,\ \ \tau_2 = \tau + \bar{\tau}_0)$$

where c and c_1 are real constants and τ_0 is any point lying outside a meridian section of the shaft. Formula (2.5) can also be written in the form

$$\Phi(r,\ z) = \int_{-r}^{r} \operatorname{Re} \chi(\xi + iz)(r^2 - \xi^2)^{3/2} d\xi = 2\int_0^{r} \operatorname{Re} \chi(\xi + iz)(r^2 - \xi^2)^{3/2} d\xi \tag{2.6}$$

The function $u(x,\ y,\ z) = w(r,\ z)e^{2i\phi}$ is, in fact, even with respect to x and y. Therefore, $\partial \omega / \partial r = 0$ at $r = 0$, and from (2.3) we have that $\operatorname{Re} \psi'(iz) = 0$, i.e. $\operatorname{Re} \chi(\xi + iz)$ is an even function with respect to ξ.

(2.6) can be considered as a Volterra integral equation of the first kind in the unknown $\operatorname{Re} \chi(\xi + iz)$ if we take r as a parameter and $\Phi(r,\ z)$ as a known function of r. This Volterra equation can be solved with the aid of a Riemann-Mellin integral transformation. In this way it can be shown that (2.5) is the general solution to Equation (1.3).

Putting $\zeta = \xi + iz$, $\tau = r + iz$, we have from (2.6) that

$$\Phi(r,\ z) = \operatorname{Re} \int_{-\tau}^{\tau} \chi(\zeta)[(\zeta - \tau)(\zeta + \bar{\tau})]^{3/2} d\zeta = \tfrac{1}{2} \operatorname{Re} \int_{l} \chi(\zeta)[(\zeta - \tau)(\zeta + \bar{\tau})]^{3/2} d\zeta \tag{2.7}$$

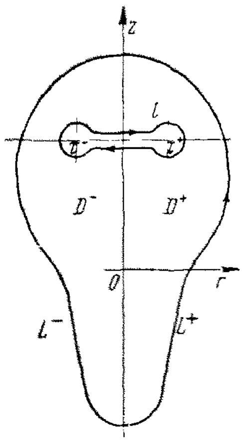

Fig. 1.

The boundary l is shown in Fig. 1. Further, it is known [3] that the function $\chi_0(r)$, which is regular within the boundary $L^+ + L^-$, where L^- is the mirror image in the z-axis, can be expressed uniquely by an integral of the Cauchy type with a real density $\phi(t)$

$$\chi(\tau) = \frac{1}{\pi i} \int\limits_{L^+ + L^-} \frac{\varphi(t)\, ds}{t - \tau} \tag{2.8}$$

Here ds is an element of arc length, t is the complex coordinate of a point on the boundary $L^+ + L^-$. From (2.7) and (2.8) we have

$$\Phi(r, z) = -\int\limits_{L+\bar{L}} \varphi(t)\,\Phi_0(t, \tau)\, ds + r^4 \,\mathrm{const} \tag{2.9}$$

where

$$\Phi_0(t, \tau) = -\mathrm{Re}\left\{\frac{1}{2\pi i}\int\limits_{\bar{l}} \frac{[(\zeta - \tau)(\zeta + \bar{\tau})]^{3/2}}{t - \zeta}\, d\zeta\right\} =$$

$$= \mathrm{Re}\,\{[(t - \tau)(t + \bar{\tau})]^{3/2} - t^3 + 3izt^2 + \tfrac{3}{2} t\,(r^2 + 2z^2)\}$$

We shall set $v(t) = \phi(-\bar{t}) - \phi(t)$, noting that

$$\Phi_0(-\bar{t}, \tau) = -\Phi_0(t, \tau)$$

and from (2.9) we have

$$\Phi(r, z) = \int\limits_{L} v(t)\,\Phi_0(t, \tau)\, ds + Cr^4 \tag{2.10}$$

The integral expression (2.10), which contains an arbitrary real function $v(t)$, is more general than (2.5): it can be used for the solution of problems on torsion of hollow shafts with the cavity formed by bodies of revolution.

Let t_0 be a point on the boundary L; then from (2.10) as $r \to t_0$ we obtain for the unknown $v(t)$ an integral equation of the first kind

$$\int\limits_{L} v(t)\,\Phi_0(t, t_0)\, ds = f(t_0) \tag{2.11}$$

Here $f(t_0)$ is a given function. An equation of the type (2.11) can be transformed to a Fredholm equation of the second kind [1]. In practice, however, it is more convenient to solve Equation (2.11) directly.

3. Torsion of a cylindrical shaft with a peripheral notch. Let us suppose that L_1^+ is the upper portion of the boundary of the notch, and that L_2^+ is the upper cylindrical portion of the boundary L. L_1^-, L_2^- are the corresponding parts of the lower portions of the boundary L.

We shall denote the torque at infinity by M, the radius of the cylindrical portion of the shaft by R, and the radius of the narrowest part of the shaft at the bottom of the notch by R_0.

$$t = t' = \xi' + i\eta' \text{ on } L_1^+, \quad t = t'' = \xi'' + i\eta' \text{ on } L_2^+$$

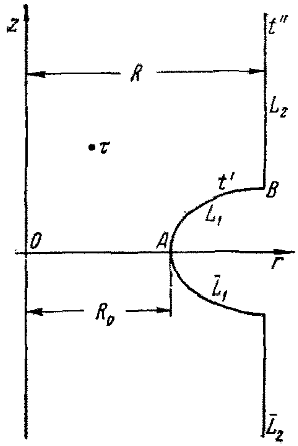

Fig. 2.

We shall seek a stress function in the form

$$\Phi(r, z) = \int_L \nu(t)\Phi_0(r, \tau)\, ds + \frac{M}{2\pi} r^4 \qquad (3.1)$$

From the condition on the boundary L it follows that

$$\Phi(r, z)\Big|_L = \text{const} = \frac{M}{2\pi} R^4$$

and in view of symmetry with respect to the r-axis, we obtain the integral equation

$$\int_{L_1^+} \nu_1(t')\Phi_*(t', \tau)\, ds' + \int_{L_2^+} \nu_2(t'')\Phi_*(t'', \tau)\, ds'' = f(\tau) \qquad (3.2)$$

Here

$$f(\tau) = \begin{cases} 0 & \text{on } L_2, \\ (M/2\pi)(R^4 - r^4) & \text{on } L_1, \end{cases} \qquad \nu(t) = \nu(\bar{t}) = \begin{cases} \nu_1(t') & \text{on } L_1 \\ \nu_2(t'') & \text{on } L_2 \end{cases}$$

$$\Phi_*(t, \tau) = \Phi_0(t, \tau) + \Phi_0(\bar{t}, \tau)$$

Obviously $\Phi_*(t, \bar{\tau}) = \Phi_*(t, \tau)$. Therefore, if we satisfy the integral equation on the upper boundaries L_1^+ and L_2^+, we automatically satisfy at the same time the condition that $\Phi(r, z) = MR^4/2\pi$ on the lower boundaries L_1^- and L_2^-.

By altering the variables we can transform Equation (3.2) into a system of integral equations of the first kind over the interval (1, 0):

$$\begin{aligned} \int_0^1 [a(x)\,\Phi_1(x,y) + b(x)\,\Phi_2(x,y)]\,dx &= F(y) \\ \int_0^1 [a(x)\,\Phi_3(x,y) + b(x)\,\Phi_4(x,y)]\,dx &= 0 \end{aligned} \tag{3.3}$$

Here

$$a(x) = \nu_1(t')\frac{ds'}{dx}, \quad b(x) = \nu_2(t'')\frac{ds''}{dx}, \qquad F(y) = \frac{M}{2\pi}(R^4 - r'(y))$$

$$\Phi_1(x,\ y) = \Phi_*(t'(x),\ \tau'(y)), \qquad \Phi_3(x,\ y) = \Phi_*(t'(x),\ \tau''(y))$$

$$\Phi_2(x,\ y) = \Phi_*(t''(x),\ \tau'(y)), \qquad \Phi_4(x,\ y) = \Phi_*(t''(x),\ \tau''(y))$$

We shall consider that with this change of variables the point $x = 0$ corresponds to the point $t' = R_0$ and $t'' = i\infty + R$.

The functions under the integral signs in (3.3) have an infinite second derivative for $x = y$. In a numerical solution of the problem, therefore, it is expedient to apply to these integrals the quadrature formula of Nikolskii [4]

$$\int_0^1 f(x)\,d(x) \approx \frac{2}{2m+1}\sum_{k=0}^{m-1} f\left(\frac{2k+2}{2m+1}\right) + \frac{2m-1}{2m+1} f(0) \tag{3.4}$$

which is most suitable for the class of functions $W^{(1)}$ $(M,\ 0.1)$ for a given natural m. Setting $y = y_k = (2k+2)/(2m+1)$ in Equations (3.3), we obtain a set of $(2m+1)$ linear algebraic equations in $(2m+1)$ unknowns $a_k = a(x_k)$, $b_k = b(x_k)$ and $a(0)$. After solving this set of equations we can find an approximate expression for $\Phi(r,\ z)$:

$$\begin{aligned} \Phi(r,\ z) \approx \frac{2}{2m+1}\sum_{k=0}^{m-1} [a_k\Phi_*(t_k',\ \tau) + b_k\Phi_*(t_k'',\ \tau)] + \\ + \frac{2m-1}{2m+1} a(0)\,\Phi_*(t_A,\ \tau) + \frac{M}{2\pi} r^4 \end{aligned} \tag{3.5}$$

The computer "Strela" at the Computer Center of Moscow University was used to give a solution for a shaft with a semicircular notch with the following data:

$$R = 1,\ M = 2\pi,\ m = 14, \quad \rho = \tfrac{1}{10},\ \tfrac{2}{10},\ \ldots,\ \tfrac{8}{10},\ \tfrac{9}{10}$$

We give below values of the coefficient k found for a number of different values of $\lambda = \rho/R$

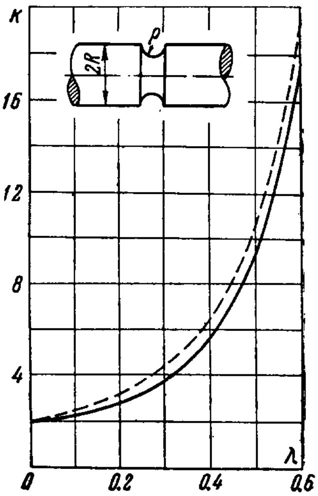

Fig. 3.

$\lambda =$	0	0.1	0.2	0.3	0.4
$k =$	2	2.292	2.861	3.888	5.780
$k^{\circ} =$	2	2.481	3.024	4.485	6.613
$\lambda =$	0.5	0.6	0.7	0.8	0.9
$k =$	9.549	18.14	42.55	145.8	1257
$k^{\circ} =$	10.67	19.53	46.60	138.9	1053

The calculation for each value of λ takes approximately three minutes on the "Strela".

The continuous curve in Fig. 3 has been drawn from these results. As a comparison the approximate values of $k = k^0$ (the broken line in Fig. 3) given by the formula of R. Sonntag

$$k^{\circ} = \frac{2}{(1-\lambda)^3(1+\lambda)}$$

have also been shown.

It will be seen from Fig. 3, that, for large values of λ, Sonntag's formula, which is the one normally used in engineering analysis, gives a decreased coefficient of stress concentration.

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